The BRST-BFV method for non-stationary systems

J. Antonio García and J. David Vergara
Instituto de Ciencias Nucleares
Universidad Nacional Autónoma de México
Apartado Postal 70-543, 04510 México, D.F.

and

Luis F. Urrutia *
Departamento de Física
Universidad Autónoma Metropolitana-Iztapalapa
Apartado Postal 55-534, 09340 México, D.F.
and
Facultad de Física
Universidad Católica de Chile
Casilla 306, Santiago 22 , Chile

Starting from an associated reparametrization-invariant action, the generalization of the BRST-BFV method for the case of non-stationary systems is constructed. The extension of the Batalin-Tyutin conversional aproach is also considered in the non-stationary case. In order to illustrate these ideas, the propagator for the time-dependent two-dimensional rotor is calculated by reformulating the problem as a system with only first-class constraints and subsequently using the BRST-BFV prescription previously obtained.

I. INTRODUCTION

The extension of the Dirac method to include constraints that depend explicitly on time was developed in [1–3], but a systematic construction of the BRST-BFV quantization procedure [4] together with the Batalin-Tyutin conversional approach [5] has not been given in detail yet [6]. We start the discussion of these problems with a brief introduction to the standard Dirac procedure for systems with time-dependent constraints.

Let us consider a system described by a $2n$-dimensional phase space (q^i, p_j), a canonical Hamiltonian $H_0(q, p, t)$, a set of first-class constraints $\{\psi_A(q, p, t)\}$, and a set of second-class constraints $\{\phi_\alpha(q, p, t)\}$. We assume that all these functions in phase space are explicitly time-dependent. The canonical description of such system is given by the action

$$S = \int (p_i \dot{q}^i - H_0 - \lambda^\alpha \phi_\alpha - \mu^A \psi_A) dt, \tag{1}$$

where λ^α and μ^A are Lagrange multipliers. We also assume that the consistency conditions

$$\frac{d\phi_\alpha}{dt} \approx 0, \quad \frac{d\psi_A}{dt} \approx 0, \tag{2}$$

are identically satisfied in order to guarantee that the Dirac procedure has been completed and no further constraints arise. The time evolution of an arbitrary time-dependent function $F(q, p, t)$ is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, \ H_T\}, \tag{3}$$

where $H_T = H_0 + \mu^A \psi_A + \lambda^\alpha \phi_\alpha$ is the total Hamiltonian. The conservation in time of the second-class constraints allow us to solve for the corresponding Lagrange multipliers leading to

---

*On sabbatical leave from Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México, D.F.

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, \ H_0\}^* + \mu^A \{F, \ \psi_A\}^* - \{F, \phi_\alpha\}C^{\alpha\beta}\frac{\partial \phi_\beta}{\partial t}, \tag{4}$$

where $\{\ \}^*$ denotes de Dirac bracket in the (q^i, p_j) phase space

$$\{\ , \ \}^* = \{\ , \ \} - \{\ , \phi_\alpha\}C^{\alpha\beta}\{\phi_\beta, \ \}, \tag{5}$$

with the matrix $C_{\alpha\beta} = \{\phi_\alpha, \ \phi_\beta\}$ been time-dependent in general. The matrix $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$. The last term in the right-hand side of Eq.(4) breaks the canonical structure of the evolution equation *i.e.* the time evolution is not any more the unfolding of a canonical transformation. Moreover, it can be readily shown that is not possible at this stage to absorb this extra term inside the Dirac bracket by adding a suitable piece to H_0 [1].

There are at least two ways of recovering the canonical structure of the evolution equations in the non-stationary case. One approach is due to Mukunda [1] and reduces to finding an apropriate change of coordinates in phase space. The basic idea is to include all second-class constraints as new coordinates. This initial set is subsequently completed by adding extra variables in such a way to end up with a set of independent invertible functions that can be considered as new coordinates (non-necessarily canonical) for the problem. The additional coordinates turn out to be the true degrees of freedom of the system provided there are no first-class constraints present. In the new coordinate patch it is possible to redefine the Hamiltonian in such a way that the equations of motion are given by the corresponding Dirac bracket only. A drawback of this method is that the adecuate non-canonical transformation is in general difficult to perform. Also, the identification of the true degrees of freedom might be completely non-trivial, or even unconvenient as it is in the case where some important symmetries turn out to be non-manifest after such identification is made.

Another approach to the same problem is given by Gitman and Tyutin [2] and consists in adding a couple of new canonically conjugated variables which are the time t together with its momentum π_t in such a a way that $\{t, \pi_t\} = 1$. Then, the evolution equation (4) can be written as a single Dirac bracket

$$\frac{dF}{dt} \approx \{F, H_0 + \pi_t + \mu^A \psi_A\}^*. \tag{6}$$

A conceptual point which must be addressed in this proposal is the question of the equivalence between the original theory and the extended one which, in principle, has two more degrees of freedom in phase space.

In this paper we clarify this equivalence by showing that the new variables t and π_t arise quite naturally by rewriting a general time-dependent system in parametrized form, as proposed in Ref. [7]. In this way, t is now an extra coordinate that evolves according to some parameter s in a canonical fashion. Such parametrized description involves an additional first-class constraint associated with the reparametrization invariance, which is just $H_0 + \pi_t$. It is precisely this constraint which supresses the additional degrees of freedom, thus getting us back to the original phase space. Following this idea, we first extend the system into an equivalent one which behaves canonically with respect to the new evolution parameter s. For this system we are able to construct the BRST-BFV propagator in the standard fashion and subsequently we proceed to integrate over the additional variables introduced, in such a way to cast the propagator in terms of the initial variables only, which include the explicit time dependence of the hamiltonian and the constraints. This method provides a natural way to construct both the BRST charge and the BRST Hamiltonian in the non-stationary case and also allows to implement the BFV prescription in this situation.

Another important question to be addressed is whether or not the FV theorem is still applicable in the non-stationary case. Our final result for the BRST-BFV propagator still contains an arbitrary fermionic gauge which is the remaining of the partial choice of the fermionic gauge in the extended canonical system, which has been fixed only in the sector of the additional variables that we want to eliminate. This final expression is subsequently used to study the validity of the FV theorem in the non-stationary case.

In this work we do not consider the quantization of systems with second class constraints through the Dirac bracket approach. Instead, we take the point of view of promoting the BRST-BFV quantization procedure to the status of a universal prescription for quantizing any constrained system. For this reason, we are also interested in extending the conversional approach of Batalin and Tyutin [5] to the non-stationary case and we also provide the necessary prescriptions.

The paper is organized as follows: in section 2 we show how the proposal of Gitman and Tyutin [2] arises naturally from a parametrized formulation of the original non-stationary system. In section 3 we use this method of dealing with time-dependent systems to show that the construction of the extended Hamiltonian in the conversional approach can be performed by starting from the corresponding expressions already known for the time-independent case; except that now the boundary condition in the recursive expressions for the Hamiltonian are changed from H_0 to $H_0 + \pi_t$. Section 4 contains the main result of the paper which is the construction of the BRST-BFV formalism for non-stationary systems, starting from the canonical prescription for the equivalent parametrized system. Finally, in section 5 we consider the simple example of a two-dimensional rotor with time-dependent radius. We use the conversional approach

to reformulate the problem as a system with first-class constrains only and we apply the BRST-BFV quantization method developped in the previous section. We calculate the quantum mechanical propagator, obtaining the well-known result.

II. REFORMULATION OF THE PROBLEM IN TERMS OF A PARAMETRIZED ACTION

In this section we consider the following reparametrization-invariant version of the action (1)

$$S = \int \left(p_i \frac{dq^i}{ds} - \frac{dt}{ds}(H_0(q,p,t) + \lambda^\alpha \phi_\alpha + \mu^A \psi_A) \right) ds, \tag{7}$$

where we demand that $\frac{dt}{ds} \neq 0$. In this approach we are promoting t to the status of a coordinate, with its corresponding canonical momentum π_t, so that the new phase space is of dimensions $2n + 2$.

Applying the standard Dirac procedure to the action (7) we obtain the primary constraints

$$\psi = \pi_t + H_0(q,p,t) \approx 0, \quad \phi_\alpha(q,p,t) \approx 0, \quad \psi_A(q,p,t) \approx 0. \tag{8}$$

The corresponding canonical Hamiltonian in zero and the total Hamiltonian $\tilde{H}_T = \tilde{\mu}(\pi_t + H_0) + \tilde{\mu}^A \psi_A + \tilde{\lambda}^\alpha \phi_\alpha$ describes the evolution with respect to the parameter s. Let us calculate the evolution of the above primary constraints, observing that none of them has an explicit s dependence. Our first result is

$$\frac{d\psi_A}{ds} \approx \tilde{\mu} \left\{ \psi_A, \ \pi_t + H_0 \right\}_{p,q,t,\pi_t} = \tilde{\mu} \left(\frac{\partial \psi_A}{\partial t} + \left\{ \psi_A, \ H_0 \right\}_{p,q,t,\pi_t} \right)$$
$$= \tilde{\mu} \left(\frac{\partial \psi_A}{\partial t} + \left\{ \psi_A, \ H_0 \right\} \right) \approx \tilde{\mu} \frac{d\psi_A}{dt,} \tag{9}$$

where the Poisson bracket without subindices is calculated with respect to the original variables q^i, p_j. The second equality in Eq.(9) follows because the involved functions do not depend on π_t. We conclude that $\frac{d\psi_A}{ds} \approx 0$ identically, in virtue of Eqs.(2). Next we calculate

$$\frac{d\phi_\alpha}{ds} \approx \tilde{\mu} \left\{ \phi_\alpha, \ \pi_t + H_0 \right\}_{q,p,t,\pi_t} + \tilde{\lambda}^\beta \left\{ \phi_\alpha, \ \phi_\beta \right\} \approx 0, \tag{10}$$

which allows us to determine the Lagrange multipliers associated to the second-class constraints

$$\tilde{\lambda}^\alpha \approx -\tilde{\mu} C^{\alpha\beta} \left\{ \phi_\beta, \ \pi_t + H_0 \right\}_{q,p,t,\pi_t}. \tag{11}$$

Finally, we are left with

$$\frac{d\psi}{ds} \approx \tilde{\lambda}^\alpha \left\{ \psi, \ \phi_\alpha \right\}_{p,q,t,\pi_t} = \tilde{\lambda}^\alpha \left\{ \pi_t + \ H_0, \phi_\alpha \right\} = \frac{1}{\tilde{\mu}} \tilde{\lambda}^\alpha C_{\alpha\beta} \tilde{\lambda}^\beta, \tag{12}$$

which is again zero in virtue of the antisymmetry of $C_{\alpha\beta}$. From the above analysis we conclude: (i) there are no secondary constraints, (ii) the constraints ψ_A and ϕ_α retain their first-class and second-class character respectively, as expected and (iii) the new constraint ψ is first-class in the corresponding Dirac brackets . It is precisely the property (iii) which guarantees the equivalence between the parametrized description (7) of the system and the original formulation (1), allowing the number of true degrees of freedom to remain the same.

Now, let us consider the evolution equation of an arbitrary function $F(q,p,t)$, which does not depend either on π_t or explicitly on the parameter s. We have

$$\frac{dF}{ds} = \left\{ F, \ \tilde{H}_T \right\}_{p,q,t,\pi_t} \approx \tilde{\mu} \{ F, \ \pi_t + H_0 \}^*_{q,p,t,\pi_t} + \tilde{\mu}^A \left\{ F, \ \psi_A \right\}_{q,p,t,\pi_t}, \tag{13}$$

where the last Poisson bracket is identical to the one calculated in the original phase space, due to the independence of both functions upon π_t. In particular we obtain

$$\frac{dt}{ds} = \tilde{\mu} \neq 0, \tag{14}$$

3

which allows us to calculate directly $\frac{dF}{dt}$ in the parametrized formulation. Comparison with Eq.(6), making the identification $\tilde{\mu}^A = \tilde{\mu}\mu^A$, leads to the result

$$\{F,\ \pi_t + H_0\}^*_{q,p,t,\pi_t} \approx \frac{\partial F}{\partial t} + \{F,\ H_0\}^* - \{F, \phi_\alpha\}C^{\alpha\beta}\frac{\partial\phi_\beta}{\partial t}, \tag{15}$$

which shows how the non-canonical contribution of the RHS is incorporated in the canonical description of the extended parametrizad system having the additional first-class constraint $\pi_t + H_0 \approx 0$. An important consequence of Eq.(15), which will be useful for our further purposes, is the statement

$$\{F,\ \pi_t + H_0\}_{q,p,t,\pi_t} \approx \frac{\partial F}{\partial t} + \{F,\ H_0\}. \tag{16}$$

III. THE BATALIN-TYUTIN CONVERSIONAL APPROACH IN THE NON-STATIONARY CASE

The conversional approach [5,7] consists in transforming all the original second-class constraints of a given system into first-class constraints in an extended phase space obtained by adding a proper set of new canonical variables. The original Hamiltonian must also be extended in such a way to preserve in time the first-class extended constraints, without generating any new constraint. For simplicity we assume that all degrees of freedom in our theory are purely bosonic.

Let us suppose that the system is described by a $2n$-dimensional phase space with coordinates (q^i, p_i), plus a Hamiltonian $H_0(q, p, t)$. We also assume for simplicity that the theory contains only second-class constraints $\phi_\alpha(q, p, t)$, which are functionally independent. Following Batalin and Tyutin [5], we introduce new degrees of freedom Φ^μ, in the same number as the original second-class constraints, with the following non-zero Poisson brackets

$$\{q^i, p_i\} = \delta^i_j\,, \qquad \{\Phi^\mu, \Phi^\nu\} = \omega^{\mu\nu}, \tag{17}$$

where the constant matrix $\omega^{\mu\nu}$ is antisymmetric and invertible. Denote by $\tau_\alpha = \tau_\alpha(p, q, \Phi, t)$ the resulting first class-constraints in the augmented phase space $(q, p) \oplus (\Phi)$, having the desired property $\{\tau_\alpha, \tau_\beta\} \approx 0$, together with the boundary conditions $\tau_\alpha(p, q, 0, t) = \phi_\alpha(p, q, t) \equiv \tau_\alpha^{(0)}(p, q, t)$. The problem of actually finding the constraints τ_α is solved constructively, in complete analogy with the time independent case, by starting from the series expansion $\tau_\alpha = \sum_{n=0}^{\infty} \tau_\alpha^{(n)}, (\tau_\alpha^{(n)} \sim \Phi^n)$, in powers of the variables Φ^μ, and demanding the first-class condition to each order in Φ. The explicit expressions that provide the solution to this problem can be found in [5], with the only difference being that most of the expanssion coefficients will become now functions of time.

The next step in the conversional approach is to construct the augmented Hamiltonian H and, in general, the augmented observables of the corresponding theory. The augmented Hamiltonian has the following properties: (i) it must reduce to the original Hamiltonian H_0 in the limit $\Phi^\mu = 0$.
(ii) it must preserve in time the first-class time-dependent constraints τ_α, so that

$$\frac{d\tau_\alpha}{dt} = \frac{\partial\tau_\alpha}{\partial t} + \{\tau_\alpha, H\}_{p,q,\Phi} \approx \{\tau_\alpha, H + \pi_t\}_{p,q,\Phi,t,\pi_t} \approx 0, \tag{18}$$

where the first weak equality in Eq.(18) is a direct consequence of Eq. (16) in the previous section. This is the basic equivalence which will permit us a direct solution of the non-stationary problem in terms of the known solution for the time independent case. In the latter situation (which we denote by a bar over the relevant quantities), let us suppose that the original second-class constraints $\phi_\alpha(q, p)$ have already been augmented to a first-class set $\{\bar{\tau}_\alpha(p, q, \Phi)\}$. The equations that determine the extended Hamiltonian in this case are

$$\left\{\bar{\tau}_\alpha, \bar{H}\right\}_{(p,q,\Phi)} \approx 0. \tag{19}$$

The solution of Eqs.(19) is given in Ref. [5] in terms of a series expansion

$$\bar{H} = \sum_{n=0}^{\infty} \bar{H}^{(n)}, \qquad \bar{H}^{(n)} \sim \Phi^n, \tag{20}$$

with the boundary condition

$$\bar{H}^{(0)} = \bar{H}_0. \tag{21}$$

4

The explicit solution is given by

$$\bar{H}^{(n+1)} = -\frac{1}{n+1}\Phi^\mu \omega_{\mu\nu} X^{\nu\rho} \bar{G}_\rho^{(n)}, \quad n \geq 0, \tag{22}$$

where $\omega_{\mu\nu}$ is the inverse of $\omega^{\mu\nu}$ defined in Eq.(17) and $X^{\mu\nu}$ is the inverse of $X_{\mu\nu}$, which is a solution of $X_{\alpha\mu} \, \omega^{\mu\nu} \, X_{\beta\nu} = -C_{\alpha\beta}$. Here $C_{\alpha\beta} = \left\{ \bar{\tau}_\alpha^{(0)}, \ \bar{\tau}_\beta^{(0)} \right\}$. The recurrence expresions for the functions $G_\rho^{(n)}$ are [5]

$$\bar{G}_\alpha^{(0)} \equiv \{\bar{\tau}_\alpha^{(0)}, \bar{H}^{(0)}\}_{(p,q)},$$

$$\bar{G}_\alpha^{(1)} \equiv \{\bar{\tau}_\alpha^{(1)}, \bar{H}^{(0)}\}_{(p,q)} + \{\bar{\tau}_\alpha^{(0)}, \bar{H}^{(1)}\}_{(p,q)} + \{\bar{\tau}_\alpha^{(2)}, \bar{H}^{(1)}\}_{(\Phi)},$$

$$\bar{G}_\alpha^{(n)} \equiv \{\bar{\tau}_\alpha^{(n)}, \bar{H}^{(0)}\}_{(p,q)} + \sum_{m=1}^n \{\bar{\tau}_\alpha^{(n-m)}, \bar{H}^{(m)}\}_{(p,q)} + \sum_{m=0}^{n-2} \{\bar{\tau}_\alpha^{(n-m)}, \bar{H}^{(m+2)}\}_{(\Phi)}$$

$$+ \{\bar{\tau}_\alpha^{(n+1)}, \bar{H}^{(1)}\}_{(\Phi)}, \tag{23}$$

where the above Poisson brackets are calculated with respect to the variables indicated in the corresponding subindices.

Now we go back to the non-stationary problem, where the corresponding condition for the extended Hamiltonian is

$$\{\tau_\alpha, H + \pi_t\}_{p,q,\Phi,t,\pi_t} \approx 0, \tag{24}$$

according to our previous Eq.(18). Let us observe that the solution of Eq.(24) in the extended phase space (q, p, Φ, t, π_t) is exactly of the same form as the solution of Eq.(19), after we consider the change $\bar{H} \to H + \pi_t$. To this end we now define

$$H + \pi_t = \sum_{n=0}^\infty H^{(n)}, \qquad H^{(n)} \sim \Phi^n. \tag{25}$$

Since the term π_t in the LHS contributes only to zero order in Φ, the boundary condition in the above equation is

$$H^{(0)} = H_0(q, p, t) + \pi_t. \tag{26}$$

In this way, the complete expressions for the functions $H^{(n)}$, $n \geq 1$, in Eq.(25) are given by the same equations (22) and (23), where the bar is now removed to indicate the explicit time dependence of the Hamiltonian and the constraints. The Poisson brackets that had the subindex (q, p) are now calculated in the extended phase space (q, p, t, π_t). Let us observe that the explicit dependence of the functions $H^{(n)}$ upon π_t appears only in $H^{(0)}$. Thus, the final Hamiltonian H will be a function of q, p, Φ and t only.

The series expansion procedure can be analogously applied to calculate the corresponding extension of any observable $A_0(q, p, t)$ of our original problem. Moreover, if we calculate the Poisson brackets of two of such augmented functions and restrict the result to $\Phi = 0$, we recover the Dirac bracket corresponding to the original problem. In other words

$$\{A, B\}_{(q,p,t,\pi_t,\Phi)}|_{\Phi=0} = \{A^{(0)}, B^{(0)}\} + \{A^{(1)}, B^{(1)}\}_{(\Phi)}$$
$$= \{A^{(0)}, B^{(0)}\} - \{A^{(0)}, \phi_\alpha\} C^{\alpha\beta} \{\phi_\beta, B^{(0)}\} \tag{27}$$
$$= \{A_0, B_0\}^*,$$

where $\{,\}^*$ is the original Dirac bracket, $A^{(0)} = A_0$ and $B^{(0)} = B_0$. We remind the reader that all Poisson brackets without subindices are calculated in the original phase space (q, p). Another interesting case is when $B = H$, $(B_0 = H_0 + \pi_t)$, which leads to

$$\frac{dA}{dt}|_{\Phi=0} = \{A, H\}_{(q,p,t,\pi_t,\Phi)}|_{\Phi=0} = \{A_0, H_0 + \pi_t\}^*_{(q,p,t,\pi_t)}$$
$$= \frac{\partial A_0}{\partial t} + \{A_0, H_0\}^* - \{A_0, \phi_\alpha\} C^{\alpha\beta} \frac{\partial \phi_\beta}{\partial t}, \tag{28}$$

which is exactly the relation (4), assuming no first-class constraints present.

IV. BRST-BFV FOR THE NON-STATIONARY CASE

Since the Batalin-Tyutin's conversional approach allows us to transform all second-class constraints into first-class constraints, including the case when we have explicit time dependence, we can apply the BRST-BFV method of quantization to arbitrary non-stationary systems. In this section we reformulate the results of [6] (and extend some results of [8]) for the non-stationary case and give an outline of the applicability of the FV theorem in this situation. For simplicity, we consider only the bosonic case.

In order to prove that the BFV method is still applicable to time-dependent systems, let us consider the reparametrized problem with the action defined by Eq. (7) but with first class constraints only (we assume that the Batalin-Tyutin conversional method of the previous section has been applied, in such a way that no second class constraints remain in the problem).

Our method consists in applying the standard BRST-BFV procedure to this canonical problem. We will subsequently show that, after choosing a convenient fermionic canonical gauge and after performing some apropriate functional integrals, we are able to obtain the expression of the BRST-BFV formulation for the original non-stationary problem given by the action (1), with first-class constraints only . From the point of view of the calculation of the evolution operator, this procedure can be considered as the inverse of the reparametrization program, whereby starting from the action (7) we obtain the propagator corresponding to the time-dependent action (1).

As we know from the discussion in section 2, reparametrization invariance introduces an additional first-class constraint ψ, in such a way that our complete set of fist-class constraints is now

$$\psi = \pi_t + H_0 \approx 0, \quad \psi_A(q, p, t) \approx 0. \tag{29}$$

From now on we introduce the index $a = (0, A)$, with the choice $\psi_0 = \psi$ and we consider the phase space of canonical variables q, p, t, π_t, which evolve according to the parameter s. Following the standard steps, we promote the Lagrange multipliers λ^a, $a = 0, 1, 2, ...m$ to dynamical variables associating to each of them a real momentum π_a, with the same Grassmann parity, such that

$$\{\pi_a, \lambda^b\} = -\delta_a^b. \tag{30}$$

These momenta are constrained to vanish in order not to change the dynamical content of the theory. Denoting by G_α, the $2m + 2$ total number of constraints (including $\pi_0 \approx 0$, $\pi_A \approx 0$) we define the vectors

$$G_\alpha = (\pi_a, \psi_a), \quad \eta^\alpha = (-i\mathcal{P}^a, C^a), \quad \mathcal{P}_\alpha = (i\bar{C}_a, \bar{\mathcal{P}}_a), \tag{31}$$

with $G_0 = (\pi_0, \psi_0 \equiv \psi)$, $G_\alpha \equiv (\pi_A, \psi_A)$ for $\alpha \neq 0$ and where $(\mathcal{P}^a, \bar{C}_a)$ together with $(C^a, \bar{\mathcal{P}}_a)$ are canonically conjugated odd Grassmann variables.

Next we construct the canonical BRST generator $\tilde{\Omega}$ in the extended phase-space $(q, p, t, \pi_t, \lambda, \pi_\lambda, \eta, \mathcal{P})$ following the standard prescription of Ref. [8]. The effective action for this system is

$$S_{eff} = \int_{s_1}^{s_2} \left(\frac{dq^i}{ds} p_i + \frac{dt}{ds} \pi_t - \lambda^a \frac{d\pi_a}{ds} + \frac{d\eta^\alpha}{ds} \mathcal{P}_\alpha - \tilde{H}_{eff} \right) ds \tag{32}$$

with

$$\tilde{H}_{eff} = \tilde{H}_{BRST} - \{\tilde{K}, \tilde{\Omega}\}, \tag{33}$$

where $\tilde{H}_{BRST}$ is identically zero because of reparametrization invariance. The evolution operator is given by

$$Z_{\tilde{K}} = \int \mathcal{D}\mu \, exp(iS_{eff}), \tag{34}$$

where $\mathcal{D}\mu$ is the Liouville measure in the above mentioned extended phase-space.

The basic idea in what follows is to make explicit the dependence of S_{eff} upon the extra canonical variables (with respect to the original problem): $t, \pi_t, \lambda^0, \pi_0, C^0, \mathcal{P}^0, \bar{C}_0, \bar{\mathcal{P}}_0$ and to perform the corresponding functional integrals in the evolution operator (34), so that we are left with the evolution operator associated with the original non-stationary problem. To this end, let us first split $\tilde{\Omega}$ in the form

$$\tilde{\Omega} = \eta^\alpha G_\alpha + \text{``more''} = \tilde{\Omega}_{min} - i\mathcal{P}^0 \pi_0 - i\mathcal{P}^A \pi_A, \tag{35}$$

where $\tilde{\Omega}_{min} = \tilde{\Omega}_{min}(t, \pi_t, q, p, C^0, \bar{\mathcal{P}}_0, C^A, \bar{\mathcal{P}}_A)$ is still nilpotent . By making explicit the dependence of $\tilde{\Omega}_{min}$ upon C^0, the new functions h and Ω_{min} are defined, such that

$$\tilde{\Omega}_{min} = C^0 h + \Omega_{min}. \tag{36}$$

A direct extension of the Theorem (6.1) of Ref. [8] allows us to prove that h is linear in π_t, *i.e.* $h = (\pi_t + H_{BRST})$ and that both H_{BRST} and Ω_{min} depend only upon the variables $t, q, p, C^A, \bar{\mathcal{P}}_A$ and not either on $\bar{\mathcal{P}}_0$ or on π_t. Let us remind the reader that our notation differs from that of of Ref. [8] : our original first-class constarints are labeled by the subindex A, while the extended constraints, which include $\psi_0 = \pi_t + H_0$ are labeled by the subindex a. Such an extension is necessary because in our case $\lambda = \pi_t$ has non-zero Poisson bracket with one of the remaining canonical variables : t. Nevertheless, the proof of the above assertions follows the same steps given in Ref. [8] for the time-independent case. The reasons are basically the following : (i) The zeroth-order dependence upon π_t comes only from $\overset{(0)}{U} = \eta^0(\pi_t + H_0(q, p, t)) + \eta^A G_A(q, p, t)$. (ii) The higher order structure functions $\overset{(n)}{U}$ can only depend on π_t through the term $\{\overset{(n)}{U}, \overset{(0)}{U}\}$. This Poisson bracket does not introduce an aditional dependence on π_t, but produces a further term proportional to $\frac{\partial}{\partial t} \overset{(n)}{U}$ which was not present in the time-independent case. (iii) Another difference that appears in our case is that the structure functions $C_{B0}^A = V_B^A$ are defined by $[\psi_A, H_0] + \frac{\partial \psi_A}{\partial t} = V_A^C \psi_C$. (iv) Again, the structure functions $\overset{(n)}{U}{}^{a_1 \ldots a_n}$ can be chosen to be zero whenever one of the upper indices a_i is equal to zero. This is a direct consequence of the independence of the ghost and implies that $\tilde{\Omega}_{min}$ is independent of $\bar{\mathcal{P}}_0$.

The nilpotency of $\tilde{\Omega}_{min}$, together with the specific dependence upon the canonical variables of the involved functions leads to the properties

$$\{\Omega, \ \Omega\} = 0, \tag{37}$$

$$\{h, \ \Omega\} = 0 = \frac{\partial \Omega}{\partial t} + \{H_{BRST}, \ \Omega\}, \tag{38}$$

where we have reintroduced the full BRST charge $\Omega = \Omega_{min} - i\mathcal{P}^A \pi_A$. The above properties, together with the final expression that we will obtain for the evolution operator (34), once we have performed the extra functional integrals, lead to the interpretation of Ω and H_{BRST} as the BRST-charge and BRST-Hamiltonian respectively, corresponding to the original non-stationary problem. Once more we emphasize that both quantities are directly obtained from the construction of the canonical BRST-charge $\tilde{\Omega}_{min}$.

Now we perform the extra functional integrals referred to above. With this purpose we choose the canonical fermionic gauge

$$\tilde{K} = \frac{i}{\epsilon} \bar{C}_0 \chi^0 - \bar{\mathcal{P}}_0 \lambda^0 + K'. \tag{39}$$

Here K' is an arbitrary fermionic gauge which does not depend either on the coordinates or the momenta with index 0 in the extended phase space, but which might depend on the parameter s. The function χ^0 will be subsequently selected to fix the gauge associated with reparametrizations. Substituting the above choice of the fermionic gauge, together with Eqs. (35), (36), in Eq. (33) leads to

$$H_{eff} = -\frac{1}{\epsilon} \chi^0 \pi_0 - \frac{i}{\epsilon} \bar{C}_0 [\chi^0, h] C^0 - \lambda^0 h - i\bar{\mathcal{P}}_0 \mathcal{P}^0 - [K', \Omega]. \tag{40}$$

Using Eq.(40) in the effective action, making the change of integration variables

$$\bar{C}_0 \rightarrow \epsilon \bar{C}_0 \qquad \pi_0 \rightarrow \epsilon \pi_0, \tag{41}$$

with Jacobian one, and choosing the gauge fixing function

$$\chi^0 = \frac{s_1 - s}{S} T - t_1 + t, \tag{42}$$

with $T \equiv t_2 - t_1$, $S \equiv s_2 - s_1$, we are able to implement the canonical gauge $\chi^0 = 0$ and to guarantee that the correct end point conditions are satisfied after taking the limit $\epsilon \rightarrow 0$ [9] . Then, the effective action can be rewritten as

$$S_{eff}^{K'} = \int_{s_1}^{s_2} ds \left(\frac{dq^i}{ds} p_i + \frac{dt}{ds} \pi_t + \frac{d\mathcal{P}^A}{ds} \bar{C}_A + \frac{dC^A}{ds} \bar{\mathcal{P}}_A - \lambda^A \frac{d\pi_A}{ds} \right.$$

$$\left. + \{K', \Omega\} + \frac{dC^0}{ds} \bar{\mathcal{P}}_0 + \pi_0 \chi^0 + i\bar{C}_0 \{\chi^0, h\} C^0 + i\bar{\mathcal{P}}_0 \mathcal{P}^0 + \lambda^0 h \right), \tag{43}$$

where $\{\chi^0, h\} = 1$. After performing the functional integrals over the ghosts $C^0, \mathcal{P}_0, \bar{C}_0, \bar{\mathcal{P}}_0$, and λ_0 we obtain a delta functional that we use to integrate over π_t. Then, we integrate π_0 to obtain a delta functional of the gauge condition, that allows us to integrate over t. The effective action reduces to

$$S_{eff}^{K'} = \int_{s_1}^{s_2} ds \left(\frac{dq^i}{ds} p_i + \frac{d\mathcal{P}^A}{ds} \bar{C}_A + \frac{dC^A}{ds} \bar{\mathcal{P}}_A - \lambda^A \frac{d\pi_A}{ds} - \frac{T}{S} H_{BRST} + \{K', \Omega\} \right), \tag{44}$$

with

$$t = \frac{s - s_1}{S} T + t_1. \tag{45}$$

If we now make a redefinition of the fermionic gauge : $K' = \frac{T}{S} K$ and use Eq.(45) to write $ds = \frac{S}{T} dt$, we obtain the following effective action for the original problem

$$S_{eff}^{K} = \int_{t_1}^{t_2} dt \left(\frac{dq^i}{dt} p_i + \frac{d\mathcal{P}^A}{dt} \bar{C}_A + \frac{dC'^A}{dt} \bar{\mathcal{P}}_A - \lambda^A \frac{d\pi_A}{dt} - H_{BRST}(q, p, t) + \{K, \Omega\} \right) \tag{46}$$

This is the main result of this paper, which amounts to the calculation of the effective action to be used in the application of the BRST-BFV method for the time-dependent case. The above expression (46) allows for the discussion of the applicability of the FV theorem to this situation. Following steps analogous to those of the proof given in theorem 9.1 of Ref. [8] for the time-independent case, we are able to verify that S_{eff} is indeed independent of the fermionic gauge K. Let us observe that expression (46) is what one would have naively expected, except maybe for the condition (38) upon H_{BRST} which enforces the statement that Ω is a conserved time-dependent charge. Since H_{BRST} is explicitly time dependent, the result that it is not a BRST-observable seems rather reasonable. It is worth noticing that Eq. (38), which defines H_{BRST} once Ω is constructed according to Eq.(37), is invariant under the change $H_{BRST} \to H_{BRST} + [K, \Omega]$ independently of the fact that H_{BRST} is not an observable.

Before closing this section we make some comments that will permit the direct calculation of Ω and H_{BRST} to be used in Eq.(46), without having to go through the reparametrized construction in each case. From Eq. (37), together with the properties of the functions involved, we conclude that the construction of the structure functions, leading to the nilpotent BRST charge Ω, is unchanged with respect to the stationary case, except for the explicit time dependence that arises now.

Nevertheless, Eq. (38) implies that some modifications appear when considering the construction of H_{BRST} in the time-dependent case. To make contact with the standard construction let us recall that Eq.(38) can be rewritten as

$$\{\Omega, H_{BRST} + \pi_t\}_{(\ldots, t, \pi_t)} = 0, \tag{47}$$

which has exactly the same form as the corresponding equation that satisfies the BRST-Hamiltonian in the time independent case, except for the (t, π_t) extension of the phase space. Since we know the solution to the former case [8], we can apply it directly to the non-stationary situation by defining

$$H_{BRST} + \pi_t = \sum_{n \geq 0} H^{(n)}, \tag{48}$$

where the only difference is again the boundary condition

$$H^{(0)} = H_0 + \pi_t. \tag{49}$$

In this sense, the situation is similar to that of the extension of the conversional approach decribed in Section 3. In the present case however, the series expansion (48) is performed in terms of the ghost and anti-ghost fields in such a way that each $H^{(n)}$ has ghost-number zero. The detailed recurrence relations that allow for the construction of the functions $H^{(n)}$ can be found in Ref. [8].

V. THE BRST-BFV QUANTIZATION OF THE TIME-DEPENDENT TWO-DIMENSIONAL ROTOR

In order to illustrate in a transparent way the general properties previously described, we discuss in this section the example of the two-dimensional rotor with time dependent radius. Following the comments at the end of Section 4, we have chosen to deal with this problem using directly the final results (34), (37), (38), and (46) obtained in the previous section, instead of starting again from a parametrized version of the system.

The Lagrangian of the system can be written as

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \lambda(r - a(t)), \tag{50}$$

where m is the mass of the particle, r, θ are polar coordinates in the plane, $a(t)$ is the time dependent radius and λ is a Lagrange multiplier. The associated momenta are $p_r = m\dot{r}$ and $p_\theta = mr^2\dot{\theta}$, which lead to the total Hamiltonian

$$H_{T0} = H_0 + \lambda(r - a) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \lambda(r - a), \tag{51}$$

together with the constraints

$$\xi_1 = r - a(t) \approx 0, \qquad \xi_2 = p_r - m\dot{a}(t) \approx 0. \tag{52}$$

The compatibility condition $\frac{d\xi_2}{dt} \approx 0$, allows us to determine the Lagrange multiplier λ as

$$\lambda = \frac{p_\theta^2}{mr^3} - m\ddot{a}. \tag{53}$$

In this way the Dirac procedure stops and we conclude that the constraints (52) are second-class and time-dependent
.

In order to carry on the BRST-BFV quantization procedure it is necessary to start from a system including only first-class constraints. To this end we apply the conversional procedure described in section 3, to the second class constraints ξ_1, ξ_2. This requires the addition of two canonically conjugated variables q and π, such that $\{q, \pi\} = 1$. A direct application of the method leads to the augmented first class constraints

$$\tau_1 = \xi_1 + q, \qquad \tau_2 = \xi_2 - \pi. \tag{54}$$

The total augmented Hamiltonian H_T can be constructed according to Eqs. (25), (26), (22) and (23). Starting from

$$H^{(0)} = H_{T0} + \pi_t = \frac{1}{2m}(p_r^2 + \frac{p_\theta^2}{r^2}) + \lambda(r - a) + \pi_t, \tag{55}$$

we calculate the first three terms in the expansion (25)

$$\begin{aligned} H^{(1)} &= -q(\frac{p_\theta^2}{mr^3} - \lambda - m\ddot{a}) - \pi(\frac{p_r}{m} - \dot{a}), \\ H^{(2)} &= \frac{1}{2m}(3q^2\frac{p_\theta^2}{r^4} + \pi^2), \\ H^{(3)} &= -\frac{6q^3}{4m}\frac{p_\theta^2}{r^5}. \end{aligned} \tag{56}$$

From these relations we can infer the form of the full augmented Hamiltonian if we make use of the binomial series for $(r + q)^{-2}$. The result is

$$H_T = \frac{1}{2m}[\pi_\eta^2 + \frac{p_\theta^2}{Q^2}] + \lambda(Q - a(t)) + m\ddot{a}(Q/2 - \eta) + \dot{a}(\pi_Q - \pi_\eta/2), \tag{57}$$

where we have introduced new canonical variables

$$\eta = \frac{1}{2}(r - q), \quad \pi_\eta = p_r - \pi, \quad Q = r + q, \quad \pi_Q = \frac{1}{2}(p_r + \pi). \tag{58}$$

We have explicitly verified that the above expression (57) does preserve in time the first-class constraints (54). The full extended Hamiltonian is $\mathcal{H}_E = \mathcal{H}_0 + \lambda^\alpha \tau_\alpha$ where $\mathcal{H}_0$ is defined by subtracting the term $\lambda \tau_1$ in Eq. (57). The corresponding augmented first order Lagrangian function is

$$L_E = -\dot{\pi}_Q Q + p_\theta \dot{\theta} + \pi_\eta \dot{\eta} - \frac{1}{2m}\left[\pi_\eta^2 + \frac{p_\theta^2}{Q^2}\right] - m\ddot{a}(Q/2 - \eta) - \dot{a}(\pi_Q - \pi_\eta/2) - \lambda(Q-a) - \sigma(\pi_\eta - m\dot{a}). \tag{59}$$

Following the standard BRST formulation, we define the vectors

$$\begin{aligned} G_\alpha &= (\pi_\lambda, \pi_\sigma, Q - a, \pi_\eta - m\dot{a}), \\ \eta^\alpha &= (-i\mathcal{P}^1, -i\mathcal{P}^2, C^1, C^2), \\ \mathcal{P}_\alpha &= (i\bar{C}_1, i\bar{C}_2, \bar{\mathcal{P}}_1, \bar{\mathcal{P}}_2), \end{aligned} \tag{60}$$

where the last two correspond to the ghosts and anti-ghosts respectively, which have the Poisson brackets $\{\eta^\alpha, \mathcal{P}_\beta\} = \{\mathcal{P}_\beta, \eta^\alpha\} = -\delta_\beta^\alpha$. The BRST charge Ω is given by

$$\Omega = -i\mathcal{P}^1 \pi_\lambda - i\mathcal{P}^2 \pi_\sigma + C^1(Q - a) + C^2(\pi_\eta - m\dot{a}). \tag{61}$$

The evolution operator is determined by the effective quantum action $Z = \int \mathcal{D}\mu \exp(iS_{eff})$, where $\mathcal{D}\mu$ is the Liouville measure corresponding to all canonical variables involved. According to our result (46) the expression for S_{eff} is

$$S_{eff} = \int_{t_1}^{t_2} dt \left(\dot{q}^i p_i - \lambda^a \dot{\pi}_a + \dot{\eta}^\alpha \mathcal{P}_\alpha - H_{eff}\right). \tag{62}$$

The effective Hamiltonian is defined by $H_{eff} = H_{BRST} - \{K, \Omega\}$, where K is the fermionic gauge fixing term. Since the extended Hamiltonian H_E already satisfies Eq.(38), we take $H_{BRST} = H_E$. Imposing the fermionic gauge condition $K = \bar{\mathcal{P}}_1 \lambda + \bar{\mathcal{P}}_2 \sigma$, the effective Hamiltonian can be written as

$$\begin{aligned} H_{eff} = {}&\frac{1}{2m}\left(\pi_\eta^2 + \frac{p_\theta^2}{Q^2}\right) + i\bar{\mathcal{P}}_1 \mathcal{P}^1 + i\bar{\mathcal{P}}_2 \mathcal{P}^2 \\ &+ \lambda(Q - a) + \sigma(\pi_\eta - m\dot{a}) + m\ddot{a}(\frac{Q}{2} - \eta) + \dot{a}(\pi_Q - \frac{\pi_\eta}{2}) \end{aligned} \tag{63}$$

and the classical effective action reads

$$\begin{aligned} S_{eff} = \int_{t_1}^{t_2} dt \Big[&p_\theta \dot{\theta} + \pi_\eta \dot{\eta} - \dot{\pi}_Q Q - \dot{\pi}_\lambda \lambda - \dot{\pi}_\sigma \sigma + \dot{\mathcal{P}}^1 \bar{C}_1 + \dot{\mathcal{P}}^2 \bar{C}_2 + \dot{C}^1 \bar{\mathcal{P}}_1 \\ &+ \dot{C}^2 \bar{\mathcal{P}}_2 - \frac{p_\theta^2}{2mQ^2} - \frac{\pi_\eta^2}{2m} - m\ddot{a}(\frac{Q}{2} - \eta) - \dot{a}(\pi_Q - \frac{\pi_\eta}{2}) - i\bar{\mathcal{P}}_1 \mathcal{P}^1 \\ &- i\bar{\mathcal{P}}_2 \mathcal{P}^2 - \lambda(Q - a) - \sigma(\pi_\eta - m\dot{a}) \Big]. \end{aligned} \tag{64}$$

Now we consider the appropriate end-point conditions for the corresponding variables. These conditions must be BRST-invariant and they should provide a unique solution for the equations of motion derived from the above effective action.

In our case, the ghosts and anti-ghosts are not coupled to the remaining variables and we obtain the equations

$$\ddot{\bar{C}}_k = 0, \quad \ddot{C}^k = 0. \tag{65}$$

Then, it is enough to choose the following end-point conditions for this sector of the problem

$$\begin{aligned} C^1(t_1) &= C^2(t_1) = C^1(t_2) = C^2(t_2) = 0, \\ \bar{C}^1(t_1) &= \bar{C}^2(t_1) = \bar{C}^1(t_2) = \bar{C}^2(t_2) = 0. \end{aligned} \tag{66}$$

The remaining equations of motion are

$$\dot{p}_\theta = 0 \qquad \dot{\theta} - \frac{p_\theta}{mQ^2} = 0,$$

$$\dot{\pi}_\eta - m\ddot{a} = 0 \qquad \dot{\eta} - \pi_\eta/m + \dot{a}/2 - \sigma = 0,$$

$$\dot{\pi}_Q - \frac{p_\theta^2}{mQ^3} + \frac{m\ddot{a}}{2} + \lambda = 0 \qquad \dot{Q} - \dot{a} = 0, \tag{67}$$

$$\dot{\lambda} = 0 \qquad \dot{\pi}_\lambda + Q - a = 0,$$

$$\dot{\sigma} = 0 \qquad \dot{\pi}_\sigma + \pi_\eta - m\dot{a} = 0.$$

In particular, they imply $\ddot{\pi}_\lambda = 0 = \ddot{\pi}_\sigma$ so that we can impose the following boundary conditions

$$\pi_\lambda(\tau_1) = \pi_\lambda(\tau_2) = \pi_\sigma(\tau_1) = \pi_\sigma(\tau_2) = 0, \tag{68}$$

in order to guarantee that $\pi_\lambda(t) = 0 = \pi_\sigma(t)$. In this way we recover the original constraints $Q - a = 0$ and $\pi_\eta - m\dot{a} = 0$ as a consequence of the equations of motion. The remaining second-order differential equations that decouple the system of equations (67) are

$$\ddot{\eta} - \frac{1}{2}\ddot{a} = 0, \qquad \ddot{\theta} + \frac{2p_\theta}{ma^3}\dot{\theta} = 0, \qquad \ddot{\pi}_Q + \frac{3p_\theta{}^2}{ma^4}\dot{a} + m\ddot{a} = 0. \tag{69}$$

Unique solutions to the above equations are obtained by fixing the end-points of the corresponding variables. Denoting by $z(t_1) = z_1$, $z(t_2) = z_2$ the corresponding fixed values, the remaining variables $p_\theta, \sigma, \lambda$ are uniquely determined in the following form

$$p_\theta = \frac{m(\theta_2 - \theta_1)}{\int_{t_1}^{t_2} a^{-2}dt}, \qquad \sigma = \frac{\eta_2 - \eta_1}{t_2 - t_1} - \frac{a_2 - a_1}{2(t_2 - t_1)},$$

$$\lambda = \frac{p_\theta^2}{m(t_2 - t_1)} \int_{t_1}^{t_2} a^{-3}dt - \frac{m(\dot{a}_2 - \dot{a}_1)}{t_2 - t_1} - \frac{\pi_{Q_2} - \pi_{Q_1}}{t_2 - t_1}. \tag{70}$$

Now we further specify the above boundary conditions. We choose

$$\eta_1 = \frac{1}{2}a(t_1) \qquad \eta_2 = \frac{1}{2}a(t_2), \tag{71}$$

which leads to $q(t) = 0$ implying also that $\sigma = 0$. Our next choice is

$$\pi_{Q_1} = \pi_{Q_2} = 0, \tag{72}$$

in such a way that Eq.(70) leads to the correct Lagrange multiplier λ in the time-independent case. We have verified that all the imposed boundary conditions are in fact BRST-invariant.

Next we calculate the integral

$$Z = \int \mathcal{D}P^1 \mathcal{D}\bar{P}_1 \mathcal{D}C^1 \mathcal{D}\bar{C}_1 \mathcal{D}P^2 \mathcal{D}\bar{P}_2 \mathcal{D}C^2 \mathcal{D}\bar{C}_2$$

$$\mathcal{D}\pi_\eta \mathcal{D}\eta \mathcal{D}p_\theta \mathcal{D}\theta \mathcal{D}\pi_Q \mathcal{D}Q \mathcal{D}\pi_\lambda \mathcal{D}\lambda \mathcal{D}\pi_\sigma \mathcal{D}\sigma \exp[iS_{eff}]. \tag{73}$$

The integration over the ghosts gives an overall factor of T^2 where $T = \tau_2 - \tau_1$, so that Z is reduced to

$$Z = T^2 \int \mathcal{D}\pi_\eta \mathcal{D}\eta \mathcal{D}p_\theta \mathcal{D}\theta \mathcal{D}\pi_Q \mathcal{D}Q \mathcal{D}\pi_\lambda \mathcal{D}\lambda \mathcal{D}\pi_\sigma \mathcal{D}\sigma$$

$$\exp\left[i \int_{\tau_1}^{\tau_2} d\tau \left(p_\theta \dot{\theta} + \pi_\eta \dot{\eta} - \dot{\pi}_Q Q - \dot{\pi}_\lambda \lambda - \dot{\pi}_\sigma \sigma - \frac{p_\theta^2}{2mQ^2} - \frac{\pi_\eta^2}{2m}\right.\right. \tag{74}$$

$$\left.\left. - m\ddot{a}(\frac{Q}{2} - \eta) - \dot{a}(\pi_Q - \frac{\pi_\eta}{2}) - \lambda(Q - a) - \sigma(\pi_\eta - m\dot{a})\right)\right]$$

The remaining functional integrations are calculated by using the general expression

11

$$\int \mathcal{D}q\mathcal{D}p \exp i \int_{t_1}^{t_2} dt(p\dot{q} + F(p;z) + q\dot{g}(t)) =$$

$$\int_{-\infty}^{+\infty} dp_0 \exp i\{p_0(q_1 - q_2) + q_1(g(t_2) - g(t_1))\} \times \tag{75}$$

$$\exp i \int_{t_1}^{t_2} dtF(p_0 + g(t) - g(t_2);z),$$

where the variable p is free at the end-points, while q is fixed by the conditions $q(t_1) = q_1$, $q(t_2) = q_2$. We have denoted by z any other variable involved and $g(t)$ is an arbitrary function of time .

The integration over $\sigma, \pi_\sigma, \lambda, \pi_\lambda$ correspond to the above case with $g = 0$ and the result is

$$Z = T^2 \int d\sigma_0 d\lambda_0 \mathcal{D}\pi_\eta \mathcal{D}\eta \mathcal{D}p_\theta \mathcal{D}\theta \mathcal{D}\pi_Q \mathcal{D}Q$$

$$\exp\left[i \int_{\tau_1}^{\tau_2} d\tau \left(p_\theta \dot{\theta} + \pi_\eta \dot{\eta} - \dot{\pi}_Q Q - \frac{p_\theta^2}{2mQ^2} - \frac{\pi_\eta^2}{2m}\right.\right. \tag{76}$$

$$\left.\left. - m\ddot{a}(\frac{Q}{2} - \eta) - \dot{a}(\pi_Q - \frac{\pi_\eta}{2}) - \lambda_0(Q - a) - \sigma_0(\pi_\eta - m\dot{a})\right)\right]$$

The remaining functional integrations over η, π_η and π_Q, Q can also be done according to the above formula . The integrations with respect to λ_0 and σ_0 contribute each with a factor of $1/T$ and produce adequate δ-functions. The final result is

$$Z = \int \mathcal{D}p_\theta \mathcal{D}\theta \exp\left[i \int_{\tau_1}^{\tau_2} d\tau \left(p_\theta \dot{\theta} - \frac{p_\theta^2}{2ma^2(\tau)} - \frac{m}{2}\dot{a}^2(\tau)\right)\right]. \tag{77}$$

The last term in the resulting expression for Z comes from the term $p_\tau^2/(2m)$ from the original variational principle. This term itself is a total time derivative and contribute to Z only with an overall phase factor. The resulting expression for Z, up to a phase factor, is [10]

$$Z = \langle\theta_2\tau_2|\theta_1\tau_1\rangle = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \exp\left[in(\theta_2 - \theta_1) - i\frac{n^2}{2m}\int_{\tau_1}^{\tau_2} d\tau \frac{1}{a^2(\tau)}\right] \tag{78}$$

In this form we recover the well known expression for the propagator of the time dependent rigid rotor.

VI. ACKNOWLEDGEMENTS

[1] N.Mukunda, Phys.Scripta **21**(1980)801.
[2] D.M.Gitman and I.V.Tyutin, *Quantization of Fields with Constraints*, Springer Verlag, Heidelberg, 1990.
[3] J. M. Evans, *Phys. Lett.* **256B** (1991) 245.
[4] C. Becchi, A. Rouet y R. Stora, *Phys. Lett.* **52B** (1974) 344; E.S. Fradkin y G.A. Vilkovisky, CERN Report TH2332 (1977); I.A. Batalin y G.A. Vilkovisky, *Phys. Lett.* **B69** (1977) 309; E.S. Fradkin y T.E. Fradkina, *Phys. Lett.* **B72** (1977) 343.
[5] I.A. Batalin and I.V. Tyutin, Int. J. of Mod. Phys. **6**(1991)3255.
[6] I.A. Batalin and S. L. Lyakovich, in *Group theoretical methods in physics*, Nova Science, 1991.
[7] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton, New Jersey, 1992.

[8] M.Henneaux, Phys.Rep.**126**(1985)1.

[9] M. Henneaux and J. David Vergara, BRST formalism and gauge invariant operators: the example of the free relativistic particle, in *Sakharov Memorial Lectures in Physics*, Ed. L.V. Keldysh and V. Ya, Nova Science, New York 1992; M. Henneaux, C. Teitelboim, and J.D. Vergara, *Nucl. Phys.* **387B** (1992) 391.

[10] H. Klinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, Word Scientific, Singapore, 1990.